

SOME ANNULAR DISC INCLUSION PROBLEMS IN ELASTICITY

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Abstract—This paper examines the problems related to the displacement and rotation of a rigid annular disc inclusion which is embedded in bonded contact with an isotropic elastic infinite space. The analysis of the inclusion problems can be reduced to the solution of sets of triple integral equations. These equations are

1. INTRODUCTION

The class of problems related to the behaviour of flexible or rigid disc shaped inclusions embedded in elastic media is of some interest to the study of multilayered elastic materials. The studies by Collins[1] and Keer[2] examine the problems of a rigid penny shaped inclusion embedded in bonded contact with an isotropic elastic solid. These studies were subsequently extended by Kocir and Sih[3] to include elliptical disc shaped rigid inclusions. The articles by Schuldt and Sih[4, 5] examine the problems related to elliptical or penny shaped inclusions embedded in elastic media. The problems related to elliptical or penny shaped inclusions and inhomogeneities embedded in elastic media are given by Mura[13], Willis[14] and Walpole[15].

This paper examines a series of axisymmetric and asymmetric problems related to an

translation of the inclusion in the x -direction. The rotational symmetric deformations are induced by the torsion of the annular disc inclusion about the z -axis. By virtue of the geometrical geometry of the annular inclusion, these problems examine completely the

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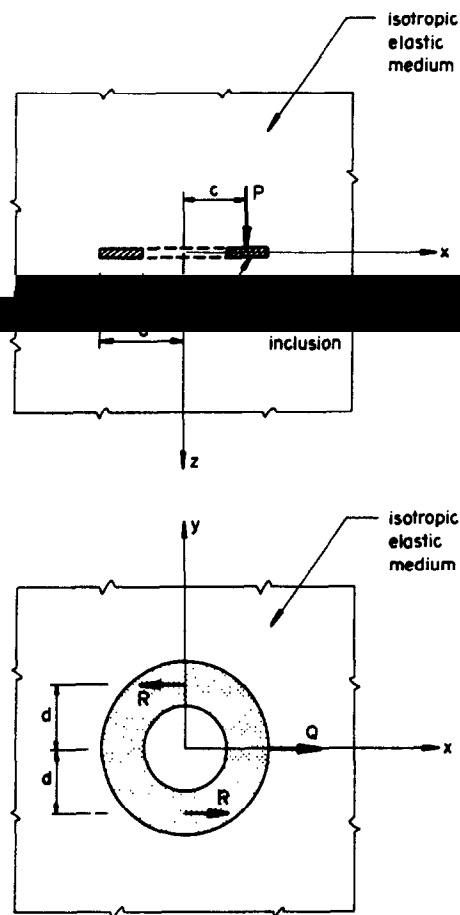


Fig. 1. Geometry of the annular disc inclusion and the resultant forces.

elastic materials. In geomechanical applications the rigid disc shaped inclusion represents the behaviour of an earth or a rock anchor which is created by the hydraulic fracture of the earth or

strengthen non-metallic or metallic matrices or increase the overall stiffness of a composite

EXAMINATION OF THE EFFECTS OF THE ANISOTROPY OF THE

In connection with the solution of the axisymmetric and asymmetric problems related to the embedded annular inclusion it is convenient to employ a formulation based on the strain potential approach of Love[21] and its extension to asymmetric problems proposed by

Somigliana-Galerkin stress function[23-24]. Proofs of the completeness of these represen-

function $\bar{\Psi}(r, \theta, z)$, i.e.:

$$\nabla^4 \Phi(r, \theta, z) = 0; \nabla^2 \Psi(r, \theta, z) = 0 \quad (1)$$

where

$$\nabla^4 = \nabla^2 \nabla^2$$

and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2} \quad (2)$$

is Laplace's operator referred to the cylindrical polar coordinate system.

We have

$$2Gu_r = -\frac{\partial^2 \Phi}{\partial r \partial z} + \frac{2}{r} \frac{\partial \Psi}{\partial \theta} \quad (3a)$$

$$2Gu_\theta = -\frac{1}{r} \frac{\partial^2 \Phi}{\partial \theta \partial z} - 2 \frac{\partial \Psi}{\partial r} \quad (3b)$$

$$2Gu_z = 2(1 - \nu) \nabla^2 \Phi - \frac{\partial^2 \Phi}{\partial z^2} \quad (3c)$$

where G and ν are the linear elastic shear modulus and Poisson's ratio respectively. Similarly the components of the stress tensor σ are given by

$$\sigma_{rr} = \frac{\partial}{\partial z} \left(\nu \nabla^2 - \frac{\partial^2}{\partial z^2} \right) \Phi + \frac{\partial}{\partial \theta} \left(\frac{2}{r} \frac{\partial}{\partial r} - \frac{2}{r^2} \right) \Psi \quad (4a)$$

$$\sigma_{\theta\theta} = \frac{\sigma}{\partial z} \left(\nu \nabla^2 - \frac{1}{r} \frac{\partial}{\partial r} - \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \right) \Phi - \frac{\sigma}{\partial \theta} \left(\frac{2}{r} \frac{\partial}{\partial r} - \frac{2}{r^2} \right) \Psi \quad (4b)$$

$$\sigma_{rz} = \frac{1}{r} \frac{\partial}{\partial \theta} \left[(1 - \nu) \nabla^2 - \frac{\partial^2}{\partial z^2} \right] \Phi - \frac{\partial^2 \Psi}{\partial r \partial z} \quad (4d)$$

$$\sigma_{zr} = \frac{\partial}{\partial r} \left[\left(\frac{1}{r} \frac{\partial}{\partial \theta} \right) \left((1 - \nu) \nabla^2 - \frac{\partial^2}{\partial z^2} \right) \Phi - \frac{1}{r} \frac{\partial^2 \Psi}{\partial \theta \partial z} \right] \quad (4e)$$

It may be noted that for axial symmetry $\Phi = \Phi(r, z)$ and $\Psi = 0$; thus the results (3) and (4) for

(i) rigid body translation δ in the x -direction (ii) rigid body rotation Ω about the y -axis
 (iii) a rigid body rotation ω about z -axis and (iv) a rotation free lateral translation δ in the

stresses, in the infinite space, about the plane $z = 0$. We may therefore restrict the analysis to a single halfspace region in which the plane $z = 0^+$ is subjected to appropriate mixed boundary conditions. From symmetry, it follows that the traction vectors at the halfspace $z = 0^+$ are

(i) For the rigid body translation in the x -direction

$$u_r(r, 0^+) = 0; r \geq 0 \quad (5a)$$

$$u_x(r, 0^+) = \delta; b \leq r \leq a \quad (5b)$$

$$\sigma_{zz}(r, 0^+) = 0; 0 < r < b. \quad (5d)$$

(ii) For the rigid body rotation about the y -axis

$$u_r(r, \theta, 0^+) = 0; r \geq 0 \quad (6a)$$

$$u_\theta(r, \theta, 0^+) = 0; r \geq 0 \quad (6b)$$

$$u_z(r, \theta, 0^+) = \Omega r \cos \theta; b \leq r \leq a \quad (6c)$$

$$\sigma_{zz}(r, \theta, 0^+) = 0; a < r < \infty \quad (6d)$$

$$\sigma_{zz}(r, \theta, 0^+) = 0; 0 < r < b. \quad (6e)$$

(iii) For the rigid body rotation about the z -axis

$$u_\theta(r, \theta, 0^+) = \omega r; b \leq r \leq a \quad (7a)$$

$$\sigma_{\theta z}(r, \theta, 0^+) = 0; a < r < \infty \quad (7b)$$

$$\sigma_{\theta z}(r, \theta, 0^+) = 0; 0 < r < b. \quad (7c)$$

(iv) For the rigid body translation along the x -direction

$$u_z(r, \theta, 0^+) = 0; r \geq 0 \quad (8a)$$

$$u_r(r, \theta, 0^+) = \delta \cos \theta; b \leq r \leq a \quad (8b)$$

$$u_\theta(r, \theta, 0^+) = -\delta \sin \theta; b \leq r \leq a \quad (8c)$$

$$\sigma_{rz} \sin \theta + \sigma_{\theta z} \cos \theta = 0; r \geq 0 \quad (8d)$$

$$\sigma_{rz} \cos \theta - \sigma_{\theta z} \sin \theta = 0; a < r < \infty \quad (8e)$$

$$\sigma_{rz} \cos \theta - \sigma_{\theta z} \sin \theta = 0; 0 < r < b. \quad (8f)$$

The boundary conditions (8d), (8e) and (8f) relate to the traction vectors which act on the plane $z = 0^+$ along the y and x directions, respectively.

solutions for Φ and Ψ take the following forms.

(i) For the rigid body translation in the z-direction

$$\Phi(r, z) = \int_0^{\infty} \xi [A(\xi) + zB(\xi)] e^{-\xi z} J_0(\xi r) d\xi \quad (9a)$$

$$\Psi(r, z) = 0. \quad (9b)$$

(ii) For the rigid body rotation about the y-axis

$$\Phi(r, \theta, z) = \left\{ \int_0^{\infty} \xi [A(\xi) + zB(\xi)] e^{-\xi z} J_1(\xi r) d\xi \right\} \cos \theta \quad (10a)$$

$$\Psi(r, \theta, z) = \left\{ \int_0^{\infty} \xi C(\xi) e^{-\xi z} J_1(\xi r) d\xi \right\} \sin \theta. \quad (10b)$$

(iii) For the rigid body rotation about the z-axis

$$\Phi(r, \theta, z) = \int_0^{\infty} \xi [A(\xi) + zB(\xi)] e^{-\xi z} J_1(\xi r) d\xi \quad (11a)$$

$$\Psi(r, \theta, z) = 0. \quad (11b)$$

(iv) For the rigid body translation along the x-direction

$$\Phi(r, \theta, z) = \left\{ \int_0^{\infty} \xi [A(\xi) + zB(\xi)] e^{-\xi z} J_1(\xi r) d\xi \right\} \cos \theta \quad (12a)$$

$$\Psi(r, \theta, z) = \left\{ \int_0^{\infty} \xi C(\xi) e^{-\xi z} J_1(\xi r) d\xi \right\} \sin \theta \quad (12b)$$

For convenience we define the n -th order Hankel operator as follows:

$$H_n[f(\xi); r] = \int_0^{\infty} \xi f(\xi) J_n(\xi r) d\xi. \quad (13)$$

the displacement and stress components given by (3a-c) and (4a-f) it can be shown that the mixed boundary conditions (5)-(8) reduce to sets of triple integral equations for an unknown function $R_n(\xi) (n = 1, 2, 3, 4)$.

(i) For the rigid body displacement of the annular disc inclusion in the z-direction we have

$$H_0[R_0(\xi); r] = 0; 0 < r < b, \quad (14a)$$

$$H_0[\xi^{-1} R_1(\xi); r] = -\frac{2\delta(1-\nu)}{(3-4\nu)}; b \leq r \leq a \quad (14b)$$

$$H_0[R_2(\xi); r] = 0; a < r < \infty. \quad (14c)$$

(ii) For the rigid rotation of the annular disc inclusion about the y -axis we have

$$H_1[\xi^{-1}R_2(\xi); r] = 0; 0 < r < b \quad (15a)$$

$$H_1[R_2(\xi); r] = -\frac{2\Omega r(1-\nu)}{(3-4\nu)}; b \leq r \leq a \quad (15b)$$

$$H_1[\xi^{-1}R_2(\xi); r] = 0; a < r < \infty. \quad (15c)$$

(iii) For the rigid rotation of the annular disc inclusion about the x -axis we have

$$H_1[R_3(\xi); r] = 0; 0 < r < b \quad (16a)$$

$$H_1[\xi^{-1}R_3(\xi); r] = \omega r; b \leq r \leq a \quad (16b)$$

(iv) For the lateral translation of the annular disc inclusion along the x -direction we have

$$H_1[R_4(\xi); r] = 0; 0 < r < b \quad (17a)$$

$$H_1[\xi^{-1}R_4(\xi); r] = -\frac{4\Delta(1-\nu)}{(7-8\nu)}; b \leq r \leq a \quad (17b)$$

$$H_1[R_4(\xi); r] = 0; a < r < \infty. \quad (17c)$$

The sets of triple integral equations defined by (14)–(17) can be solved by employing a variety of approximate techniques. Detailed descriptions of these methods are given by Williams [20], Cocks [26], Trentor [27], Collins [29] and Jain and Kanwal [29]. Complete accounts of these

the method of solution proposed by Williams [20]. In its general form, the triple system can be written as

$$H_n[R(\xi); r] = 0; 0 < r < b \quad (18)$$

$$H_n[\xi^{-1}R(\xi); r] = f(r); b \leq r \leq a \quad (19)$$

$$H_n[R(\xi); r] = 0; a < r < \infty. \quad (20)$$

We assume that the function $R(\xi)$ can be written in the form

$$H_n[R(\xi); r] = g(r); b < r < a. \quad (21)$$

From the Hankel inversion theorem we have

$$R(\xi) = \int_a^b r g(r) J_n(\xi r) dr. \quad (22)$$

Using this result in (19) we obtain

$$\int_a^b u g(u) K_0(u, r) du = f(r); b \leq r \leq a \quad (23)$$

where

$$K_0(u, r) = u \int_0^\infty J_n(\xi r) J_n(\xi u) d\xi. \quad (24)$$

We define the functions $g_1(u)$ and $g_2(u)$ such that

$$g_1(u) + g_2(u) = \begin{cases} 0 & ; 0 \leq r < b \\ g(u) & ; b \leq r \leq a \\ 0 & ; a < r < \infty \end{cases} \quad (25)$$

and assume that $f(r)$ admits expansions of the form

$$f_1(r) = \sum_{n=0}^{\infty} a_n r^n; 0 < r < a \quad (26)$$

$$f_2(r) = \sum_{n=-\infty}^{\infty} a_n r^n; b < r < \infty. \quad (27)$$

From the representations (24)–(27) it follows that the integral equation (23) reduces to two integral equations

$$\int_0^{\infty} u K_0(u, r) g_1(u) du = f_1(r); 0 < r < a \quad (28)$$

$$\int_0^{\infty} u K_0(u, r) g_2(u) du = f_2(r); b < r < \infty. \quad (29)$$

By making use of the identities (11), (12) given in the appendix it can be shown that

$$\int_1^{\infty} t K_0(r, t) g(t) dt = 4r^{-n} \int_0^r \frac{s^{2n} ds}{(r^2 - s^2)^{1/2}} \int_s^{\infty} \frac{t^{1-n} g(t) dt}{(t^2 - s^2)^{1/2}}; 0 < r < \infty \quad (30)$$

$$\int_0^{\infty} t K_0(r, t) g(t) dt = 4r^n \int_r^{\infty} \frac{s^{-2n} ds}{(r^2 - s^2)^{1/2}} \int_0^s \frac{t^{1+n} g(t) dt}{(s^2 - t^2)^{1/2}}; 0 < r < \infty. \quad (31)$$

Using these results, the integral equations (28) and (29) can be expressed in the form

$$4r^{-n} \int_s^r \frac{s^{2n} ds}{(r^2 - s^2)^{1/2}} \int_s^{\infty} \frac{t^{1-n} g_1(t) dt}{(t^2 - s^2)^{1/2}} = f_1(r); 0 < r < a \quad (32)$$

$$4r^n \int_r^{\infty} \frac{s^{-2n} ds}{(s^2 - r^2)^{1/2}} \int_0^s \frac{t^{1+n} g_2(t) dt}{(s^2 - t^2)^{1/2}} = f_2(r); b < r < \infty. \quad (33)$$

The next step is to define unknown functions $S_i(r)$, $T_i(r)$ and $C_i(r)$ ($i = 1, 2$) such that

$$\int_0^{\infty} t^{1-n} g_1(t) dt = S_1(r); 0 < r < a \quad (34)$$

$$r^{-n} \int_0^r \frac{t^{1+n} g_2(t) dt}{(r^2 - t^2)^{1/2}} = \begin{cases} -T_2(r); 0 < r < b \\ S_2(r); b < r < \infty \end{cases} \quad (35)$$

$$4r^{-n} \int_0^r \frac{s^n C_1(s) ds}{(r^2 - s^2)^{1/2}} = f_1(r); 0 < r < a \quad (36)$$

$$4r^n \int_r^{\infty} \frac{s^{-n} C_2(s) ds}{(s^2 - r^2)^{1/2}} = f_2(r); b < r < \infty \quad (37)$$

$$S_i(r) = C_i(r). \quad (38)$$

The four integral equations (35)–(38) can be inverted to give

$$g_2(t) = \frac{2}{\pi} t^{-n-1} \frac{d}{dt} \left[- \int_0^b \frac{u^{n+1} T_2(u) du}{(t^2 - u^2)^{1/2}} + \int_b^1 \frac{u^{n+1} S_2(u) du}{(t^2 - u^2)^{1/2}} \right] \quad (40)$$

$$C_1(r) = \frac{1}{2\pi r^n} \frac{d}{dr} \int_0^r \frac{u^{n+1} f_1(u) du}{(r^2 - u^2)^{1/2}}; \quad 0 < r < a \quad (41)$$

$$C_2(r) = - \frac{r^n}{2\pi} \frac{d}{dr} \int_r^\infty \frac{u^{1-n} f_2(u) du}{(u^2 - r^2)^{1/2}}; \quad b < r < \infty. \quad (42)$$

Substituting the values of $g_1(t)$ and $g_2(t)$ given by (39) and (40) into (24) and (25) (the

Fredholm integral equations of the second kind which take the forms

$$T_1(r) = l_1(r) + \frac{n!}{r^n \sqrt{(\pi \Gamma)(n + (3/2))}} \int_0^b \frac{u^{n+1} T_2(u) {}_2F_1((1/2), n; n + (3/2); (u^2/r^2)) du}{(r^2 - u^2)}; \quad a < r < \infty \quad (43)$$

where ${}_2F_1$ is a hypergeometric function and $l_1(r)$ and $l_2(r)$ are given by

$$l_1(r) = - \frac{2}{\pi r^n} \int_0^r \frac{t^{2n} dt}{(r^2 - t^2)^{1/2}} \frac{d}{dt} \int_1^a \frac{u^{1-n} S_1(u) du}{(u^2 - t^2)^{1/2}} \quad (45)$$

$$l_2(r) = 2r^n \int_r^\infty \frac{t^{-2n} dt}{(t^2 - r^2)^{1/2}} \frac{d}{dt} \int_0^b \frac{u^{n+1} S_2(u) du}{(t^2 - u^2)^{1/2}} \quad (46)$$

The integral equations (43) and (44) can be solved, by using iterative techniques, to yield expressions for $T_i(r)$; these in turn can be used in (39) and (40) to generate the expressions for $g_i(t)$. Specific results derived from the method are outlined below.

(i) For example, for the rigid body displacement of the disc inclusion in the axial direction we have

$$n = 0; f(r) = A = \text{const.}; f_1(r) = A; f_2(r) = 0. \quad (47)$$

From (38), (41) and (42) we have

$$C_1 = S_1 = \frac{A}{2\pi}; C_2 = S_2 = 0. \quad (48)$$

Making use of (43)–(46) we find that

$$l_1(br) = \frac{1}{\pi^2} \left[\lambda r + \frac{\lambda^3 r^3}{3} + \frac{\lambda^5 r^5}{5} + \frac{\lambda^7 r^7}{7} + O(\lambda^9) \right] \quad (49)$$

$$l_2(ar) = 0$$

$\lambda = b/a$ and $O(\lambda^n)$ is the Landau symbol.

By iteration we obtain from (43) and (44) the following expressions for $T_i(r)$:

$$T_1(ar) = \frac{2\lambda}{\pi^2} \left[\frac{1}{r^2} \left(\frac{\lambda}{3} + \frac{\lambda}{15} + \frac{4\lambda}{27\pi^2} + \frac{\lambda}{35} + \frac{22\lambda}{675\pi^2} \right) \right. \\ \left. + \frac{1}{r^4} \left(\frac{\lambda^3}{5} + \frac{\lambda^5}{21} + \frac{4\lambda^6}{45\pi^2} \right) + \frac{\lambda^5}{7\pi^6} + O(\lambda^7) \right]; \quad 1 < r < \infty \quad (50)$$

$$T_2(br) = \frac{1}{\pi^2} \left[\lambda r + \frac{\lambda^3 r^3}{3} + \frac{\lambda^5 r^5}{5} + \frac{\lambda^7 r^7}{7} + \frac{4}{\pi^2} \left\{ \left(\frac{\lambda^4}{9} + \frac{14\lambda^6}{225} + \frac{4\lambda^7}{81\pi^2} + \frac{29\lambda^8}{735} \right) r \right. \right. \\ \left. \left. + \left(\frac{\lambda^6}{15} + \frac{22\lambda^8}{525} \right) r^3 + \frac{\lambda^2 r^5}{21} \right\} + O(\lambda^9) \right]; 0 < r < 1. \quad (51)$$

These results can be used to develop the relevant expression for $g(t) (= g_1(t) + g_2(t))$.

(ii) For the rigid rotation of the disc inclusion about the x -axis $\alpha = 1$, $f(\alpha) = D_1$, $f(\alpha) = D_2$.

where B is a constant

$$C_1(r) = S_1(r) = 2Br; C_2(r) = S_2(r) = 0.$$

The corresponding expressions for $T_1(ar)$ and $T_2(br)$ take the forms:

$$T_1(ar) = \frac{32Ba\lambda^5}{45\pi^2} \left[\frac{1}{r^3} \left(1 + \frac{2\lambda^2}{7} \right) + \frac{6\lambda^2}{7r^5} + O(\lambda^4) \right]; 1 < r < \infty \quad (52)$$

$$T_2(br) = \frac{8Ba\lambda^5}{45\pi^2} \left[\lambda^2 r^2 + \frac{2\lambda^4 r^4}{7} + O(\lambda^6) \right]; 0 < r < 1 \quad (53)$$

Similar results can be derived for the problems which relate to rotation of the disc inclusion about the y -axis and the translation of the inclusion along the x -axis.

applications.

(i) Rigid body translation in the z -direction

Referring to Fig. 1, we note that the eccentrically applied load P can be visualized as a point force P acting at a distance δ from the z -axis. Considering the axisymmetric problem we have

$$P = 2\pi \int_b^a r [\sigma_{zz}(r, 0^-) - \sigma_{zz}(r, 0^+)] dr \quad (54)$$

where $\sigma_{zz}(r, 0^+)$ and $\sigma_{zz}(r, 0^-)$ refer to the normal interface stresses which act on the faces of the inclusion.

Considering (47) we can set $A = -2\delta(1-\nu)/(3-4\nu)$; consequently (55) yields

$$P = \frac{64(1-\nu)G\delta}{(3-4\nu)} \left[1 - \frac{4\lambda^3}{3\pi^2} - \frac{9\lambda^5}{15\pi^2} - \frac{16\lambda^6}{27\pi^4} - \frac{92\lambda^7}{315\pi^2} - \frac{448\lambda^8}{675\pi^4} + O(\lambda^9) \right]. \quad (56)$$

We note that as $\lambda \rightarrow 0$, (56) reduces to the classical result for the solid disc inclusion derived by Collins[1], Kanwal and Sharma[19] and Selvadurai[18].

(ii) *Rigid body rotation about the y-axis*

The resultant moment $M_0 = Pc$ is given by

$$M_0 = \pi \int_b^a r [\sigma_{zz}(r, \theta, 0^+) - \sigma_{zz}(r, \theta, 0^-)] r dr. \quad (57)$$

Again $\sigma_{zz}(r, \theta, 0^+) = -\sigma_{zz}(r, \theta, 0^-)$ and

$$M_0 = -4\pi G \int_b^a r^2 g(r) dr. \quad (58)$$

$$M_0 = -4\pi G \Omega a^3 \left[1 - \frac{16\lambda^5}{15\pi^2} - \frac{64\lambda^7}{105\pi^2} + O(\lambda^9) \right] \quad (59)$$

Again as $\lambda \rightarrow 0$ (59) reduces to the result for the solid inclusion given by Selvadurai[7]

(iii) *Rigid body rotation about the z-axis*

The forces R act in the plane of the disc inclusion. These forces are equivalent to a resultant torque $T (= 2Rd)$ which acts about the z-axis. The magnitude of T is given by

$$T = 2\pi \int_b^a r^2 [\sigma_{\theta z}(r, 0^+) + \sigma_{\theta z}(r, 0^-)] dr \quad (60)$$

Using the results derived in the previous sections it can be shown that

$$T = \frac{32Ga^3\omega}{3} \left[1 - \frac{16\lambda^5}{15\pi^2} - \frac{64\lambda^7}{105\pi^2} + O(\lambda^9) \right]. \quad (61)$$

The result (61) is in agreement with analogous results derived by Collins[28] for the Reissner-Sagoci problem for an annular punch.

(iv) *Rigid body translation along the x-axis*

The application of the force Q causes a rigid body translation (Δ) of the annular disc inclusion along the x-direction.

$$Q = \int_b^a \int_0^{2\pi} [T_x(r, \theta, 0^+) + T_x(r, \theta, 0^-)] r dr d\theta. \quad (62)$$

The load-displacement relationship takes the form

$$Q = \frac{64(1-\nu)Ga\Delta}{(7-8\nu)} \left[1 - \frac{4\lambda^3}{3\pi^2} - \frac{9\lambda^5}{15\pi^2} - \frac{16\lambda^6}{27\pi^4} - \frac{92\lambda^7}{315\pi^2} - \frac{448\lambda^8}{675\pi^4} + O(\lambda^9) \right]. \quad (63)$$

As $\lambda \rightarrow 0$, (63) reduces to the results given by Keer[2], Kassir and Sih[3] and Selvadurai[9] for the lateral translation of the embedded solid disc inclusion.

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APPENDIX A

$$J_n(pr) = \left(\frac{2p}{\pi}\right)^{1/2} \frac{1}{r^n} \int_0^r \frac{J_{n-1/2}(ps)s^{n+(1/2)} ds}{(r^2-s^2)^{1/2}} \quad (A1)$$

$$J_n(pr) = \left(\frac{2p}{\pi}\right)^{1/2} r^n \int_r^\infty \frac{J_{n+1/2}(ps)s^{-n+(1/2)} ds}{(s^2-r^2)^{1/2}} \quad (A2)$$

$$\int_0^\infty p J_{n+1/2}(ps) J_{n+1/2}(pt) dp = \frac{\delta^*(s-t)}{(st)^{1/2}} \quad (A3)$$

where δ^* is the Dirac delta function.