

McGill University  
Department of Mathematics and Statistics

Ph.D. preliminary examination, PART A

PURE MATHEMATICS  
Paper BETA

16 May, 2014  
13:00 { 17:00

**INSTRUCTIONS:**

(i) This paper consists of the three modules (1) Algebra, (2) Analysis, and (3) Geometry & Topology, each of which comprises 4 questions. You should answer 7 questions with at least 2 from each module. If you answer more than 7 questions, then clearly identify which 7 questions should be graded.

(ii) Pay careful attention to the exposition. Make an effort to ensure that your arguments are complete. The results you use should be quoted in full.

This exam comprises this cover and 3 pages of questions.

## Algebra Module

**[ALG. 1]** Prove that for  $n \geq 5$ , the alternating group  $A_n$  is the only non-trivial normal subgroup of the symmetric group on  $n$  elements,  $S_n$ .

**[ALG. 2]** Let  $G$  be a group of order 385. Show that the 7-Sylow subgroup of  $G$  is contained in the centre of  $G$  and that the 11-Sylow subgroup is normal.

**[ALG. 3]** Let  $F$  be a field. Consider a category  $\mathbf{C}$  whose objects are pairs  $(V; T)$  where  $V$  is an  $F$ -vector space and  $T: V \rightarrow V$  is an  $F$ -linear transformation.

- Explain how to define morphisms in  $\mathbf{C}$  such that  $\mathbf{C}$  becomes a category equivalent to the category of left  $F[x]$ -modules.
- Given a pair  $(V; T)$  what is its torsion submodule under the equivalence in part (a)? Consider the particular example where  $V = F(a_1; a_2; a_3; \dots)$ ,  $a_i \in F$  and  $T((a_1; a_2; a_3; \dots)) = (a_2; a_3; a_4; \dots)$ .
- Show that any finitely supported vector, i.e. a vector such that  $a_n = 0$  for  $n \geq 0$  is torsion.
- Show that the torsion of  $V$  contains other vectors.

**[ALG. 4]** Let  $F$  be a field of characteristic different from 2, and let  $F[x]$  the ring of polynomials in one variable over  $F$ . Let  $f(x) \in F[x]$  be a non-constant separable polynomial of degree  $n$ . Write  $f(x) = \prod_{i=1}^n (x - \alpha_i)$  in some splitting field of  $f$ . Define the discriminant of  $f$  to be

$$D(f) = \prod_{i < j} (\alpha_i - \alpha_j)^2$$

- Prove that the discriminant of  $f$  lies in  $F$  and that the Galois group of  $f$ , viewed as a subgroup of  $S_n$  via its action on the roots of  $f$ , is contained in  $A_n$  if and only if  $D(f)$  is a square in  $F$ .
- Using that the discriminant of a cubic  $f(x) = x^3 + ax + b$  is given by  $-4a^3 - 27b^2$ , prove that if  $F$  is a finite field then for every  $a, b \in F$  such that  $x^3 + ax + b$  is irreducible over  $F$ , the quantity  $-4a^3 - 27b^2$  is a square in  $F$ .

## Analysis Module

[AN. 1] Given the measure space  $(X, \mathcal{A}, \mu)$  with  $\mu(X) = 1$ , suppose that there are  $\mathcal{A}$ -measurable subsets  $E_n \subset X$ ;  $n = 1, 2, \dots$ , whose complements  $X \setminus E_n$  are disjoint. Prove that

$$\lim_{N \rightarrow \infty} \mu\left(\bigcap_{n=1}^N (E_n)\right) = \prod_{n=1}^{\infty} \mu(E_n):$$

(Hint: First prove by induction that if the  $y_k$  are  $\in [0, 1]$  and  $y_1 + y_2 + \dots + y_n = 1$ , then  $(1 - y_1)(1 - y_2) \dots (1 - y_n) \geq 1 - y_1 - y_2 - \dots - y_n$  for  $1 \leq n$ .)

[AN. 2] Show that

$$\lim_{n \rightarrow \infty} \int_0^n \frac{x^n}{n} dx$$

exists and find that limit. (Hint: First verify, e.g., using differential calculus, that  $1 - t = e^{-t}$  for  $0 \leq t \leq 1$ .)

[AN. 3] Let  $p > 1$ ,  $f(x) \in L_p(\mathbb{R})$ , and  $\{a_n\} \in l_q(\mathbb{Z})$ , where  $\frac{1}{p} + \frac{1}{q} = 1$ . Show that the series  $\sum_{n \in \mathbb{Z}} a_n f(x - n)$  converges absolutely for almost all  $x$  to a sum  $F(x)$  with  $\int_{\mathbb{R}} |F(x)|^p dx < \infty$ . Estimate the last integral in terms of  $\sum_{n \in \mathbb{Z}} |a_n|^q$  and  $\|f\|_p$ .

[AN. 4]

(a) Compute

$$\frac{1}{2} \int_{-\infty}^{\infty} e^{y|t|} \operatorname{sgn}(t) e^{ix} dt;$$

where  $y > 0$ ;  $x \in \mathbb{R}$ .

(b) Hence show that for  $f \in L_1(\mathbb{R}) \cap L_2(\mathbb{R})$ ,

$$\frac{1}{2} \int_{-\infty}^{\infty} \frac{x-t}{(x-t)^2 + y^2} f(t) dt = \frac{i}{2} \int_{-\infty}^{\infty} e^{y|t|} \hat{f}(x) e^{ix} dx$$

(when  $y > 0$  and  $x \in \mathbb{R}$ ).

(c) Conclude that if

$$\hat{f}(x) = \frac{1}{2} \int_{-\infty}^{\infty} \frac{(x-t)}{(x-t)^2 + y^2} f(t) dt;$$

we have  $\| \hat{f} \|_2 \leq \|f\|_2$ .

## Geometry and Topology Module

### [GT. 1]

Let  $X$  be the topological space whose points are  $fa; b; x; yg$  and whose topology is:

$$\{fa; b; x; yg; fa; x; bg; fa; y; bg; fa; bg; fag; fbg; fgg\}$$

- Prove that  $X$  is path connected.
- What is  $\pi_1(X; a)$ ?
- Describe the universal cover  $\tilde{X}$  of  $X$ , by indicating a basis for the topology of  $\tilde{X}$ .
- Explain why  $X$  does not have the same homotopy type as the circle  $S^1$ .

### [GT. 2]

Let  $M$  be a moebius strip with boundary, and let  $p \in M$  be a point in its interior. Let  $N = M - \text{fp}g$ .

- Draw a connected degree two covering space  $\tilde{N}_1$  of  $N$  with the property that  $p \in \tilde{N}_1$  consists of exactly one circle.
- Draw an orientable degree two covering space  $\tilde{N}_2$  of  $N$ .
- Find three embedded closed curves in  $N$  that are not homotopic to each other.
- Identify four distinct compact surfaces that are not homeomorphic to each other, but have the same homotopy type as  $N$ .

### [GT. 3]

- Show that a manifold is orientable if and only if the complement of the zero-section in the top exterior power of its tangent bundle has two components.
- One can produce the real projective plane in two different ways: as a quotient of the standard two-sphere in  $\mathbb{R}^3$  by the involution  $p \sim -p$ , and by glueing a disk to the boundary of the Moebius strip. Is the projective plane orientable? Compute its fundamental group, and its deRham cohomology.

### [GT. 4]

- Consider the three vector fields on  $\mathbb{R}^3$ :

$$X = y \frac{\partial}{\partial z} - z \frac{\partial}{\partial y}; \quad Y = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z}; \quad Z = x \frac{\partial}{\partial y} - y \frac{\partial}{\partial x};$$

Show that they form a Lie algebra.

- This Lie algebra is the Lie algebra of  $SO(3)$ . As such,  $X; Y; Z$  can be thought of as left-invariant vector fields on the group  $SO(3)$ , trivializing its tangent bundle. Let  $\theta_X; \theta_Y; \theta_Z$  be the dual basis of left-invariant one forms:  $\theta_X(X) = 1; \theta_X(Y) = 0$ ; etc.. Compute  $L_X(\theta_X); L_Y(\theta_X); L_Z(\theta_X)$ .
- Using part (b) or otherwise, compute  $d\theta_X$  as a linear combination of exterior products of  $\theta_X; \theta_Y; \theta_Z$ .